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Asymmetric directed percolation on the square lattice

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Abstract. We consider directed percolation on the square lattice, with probability $p_H(p_V)$ for the horizontal (vertical) bonds to be unbroken. For $p_H = 1-\varepsilon$ (ε small) and $p_V > \sigma(\varepsilon)$ the percolating cluster is asymptotically within a cone $\phi_+ < \phi < \phi_+$. We calculate $\phi_\pm$ and the fluctuations of the boundaries at $\phi = \phi_\pm$ as power series in ε, up to terms $\sim\varepsilon^2$, showing that the transverse spread of the percolating cluster is randomwalk-like. For any given p_H we also calculate the percolation threshold $p_{V,c}(\varepsilon)$, defined by $\phi_+ = \phi_-$ at $p_V = p_{V,c}$.

1. Introduction

There exist a large number of problems, ranging from high-energy physics (Moshe 1978) via astronomy (Schulmann and Seiden 1982) and chemistry (Schlögl 1972) to epidemiology (Griffeath 1979), which only recently have been recognised (Grassberger and de la Torre 1979, Cardy and Sugar 1980) to be related to directed percolation. (Broadbent and Hammersley 1957, Blease 1977, Obukhov 1980, Kertész and Viczek 1980, Dhar and Barma 1981, Kinzel and Yeomans 1981).‡

Let us for the moment concentrate on the specific problem of directed bond percolation on a square lattice, with probability p_H for horizontal bonds to be unbroken and p_V for vertical ones (see figure 1(a); other cases will be discussed later).

For the symmetric case $p_H = p_V$, the critical properties are known in considerable detail (Moshe 1978, Grassberger and de la Torre 1979, Cardy and Sugar 1980, Broadbent and Hammersley 1957, Blease 1977, Obukhov 1980, Kertész and Viczek 1980, Dhar and Barma 1981, Kinzel and Yeomans 1981). In particular, percolation occurs at $p_c \cong 0.645$, and the percolating cluster (for $p > p_c$) is essentially confined to a cone of width $\Delta\phi \sim (p-p_c)^{0.63}$ around the diagonal (see figure 2(a)). At the edges $\phi = 45° \pm \Delta\phi/2$ of this cone, the density of sites connected to the origin decays like an error function with a width increasing as $\sqrt{R}$ (R being the distance from the origin) (Grassberger and de la Torre 1979). This suggests that the transverse spread of the percolating cluster with increasing R is essentially random-walk-like.

The asymmetric case $p_H > p_V$ was first studied by Domany and Kinzel (1981). In this case we expect the percolating cluster, for $p_V > p_{V,c}(p_H)$, to be confined to a cone $\phi_- < \phi < \phi_+$ (see figure 2(a)). The critical point $p_{V,c}$ is defined by $\phi_+ = \phi_-$ at $p_V = p_{V,c}$. Somewhat arbitrarily, Domany and Kinzel defined another 'percolation threshold p_{DK} by $\phi_+(p_V = p_{DK}) = 45°$, i.e. for given p_H, p_{DK} is the threshold of p_V above which

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‡ Percolation was not mentioned in Grassberger and de la Torre (1979), but the fact that reggeon theory is essentially a percolation problem has already been observed in Grassberger (1977).

the diagonal is within the percolating cluster. More generally, for any angle ϕ ($0<\phi<90°$) and for sufficiently large p_H, we can define thresholds $p_\phi^{(\pm)}$ analogous to the Domany–Kinzel threshold by demanding $\phi_\pm(p_V = p_\phi^{(\pm)}) = \phi$. This is illustrated in figure 3, where p_{DK} and $p_{22.5°}^{(\pm)}$ are drawn together with $p_{V,c}$. Domany and Kinzel found that, for $p_V \neq p_H$, the behaviour near $p_V \simeq p_{DK}$ is different from the symmetric case. This should not surprise us, as it is the behaviour near $p_V \simeq p_{V,c}$ which should be universal.

For the case $p_H = 1$ they were able to solve the model completely, and it was subsequently pointed out by Wu and Stanley (1982) that this case is exactly a random-walk model.

In the present paper we shall consider the case where p_H is close to 1. More precisely, we shall put $p_H = 1-\varepsilon$ and calculate $\phi_\pm$ as power expansions in ε. In addition, we shall calculate the fluctuations of the boundaries at ϕ_+ and ϕ_-, verifying the random-walk-type behaviour up to $O(\varepsilon^2)$. From this we calculate p_{DK} perturbatively, in very good agreement with the result of Domany and Kinzel (1981).

An important property of this model is its symmetry under the exchange $\phi \to 90°-\phi$, $p_V \leftrightarrow p_H$, corresponding to a reflection about the diagonal in figure 1(*a*). This implies

$$\phi_-(p_V, p_H) = 90° - \phi_+(p_H, p_V). \tag{1}$$

In the domain $p_V \simeq p_H \simeq 1$ we can calculate both sides independently, providing a very welcome test.

In order to calculate $p_{V,c}$, we have to put $\phi_+ = \phi_-$. As we shall see, inserting here the values of ϕ_- computed directly does not produce reliable results. However, using equation (1) to compute ϕ_- yields values of $p_{V,c}$ which converge rapidly with increasing order of ε. Also, the order ε^2 result for the symmetric case $p_H = p_V$ agrees nicely with Kinzel and Yeomans (1981).

2. The perturbation expansion

Let us first redraw figure 1(*a*) and 2(*a*) in a skew coordinate system such that the resulting figures are 1(*b*) and 2(*b*) respectively. The advantage of this new representation is that all points connected by the same number n of bonds to the origin occupy now one column of a square lattice. Let us denote by 'i' the other coordinate of the lattice.

It is most natural to regard n as a time variable and i as a space coordinate. A particular realisation would then be an epidemic process on a $(1+1)$-dimensional lattice, where each site can infect its upper neighbour with rate p_V and can recover (without immunisation) with rate ε. The infinite cluster comprises then all infected space–time points, starting with one or more infected sites at time $n=0$. The 'front' of the infinite cluster at time n is defined as the point with maximal i and the 'trailing edge' by the point with minimal i.

Let us denote by $P_n^{(+)}(i)$ ($P_n^{(-)}(i)$) the probability that the front (trailing edge) in the nth column is at row i. Next we attribute to each lattice site (i, n) a variable $X_{i,n}$ such that

$$X_{i,n} = \begin{cases} 1 & \text{if site } (i,n) \text{ belongs to the infinite cluster} \\ 0 & \text{if it does not.} \end{cases} \tag{2}$$

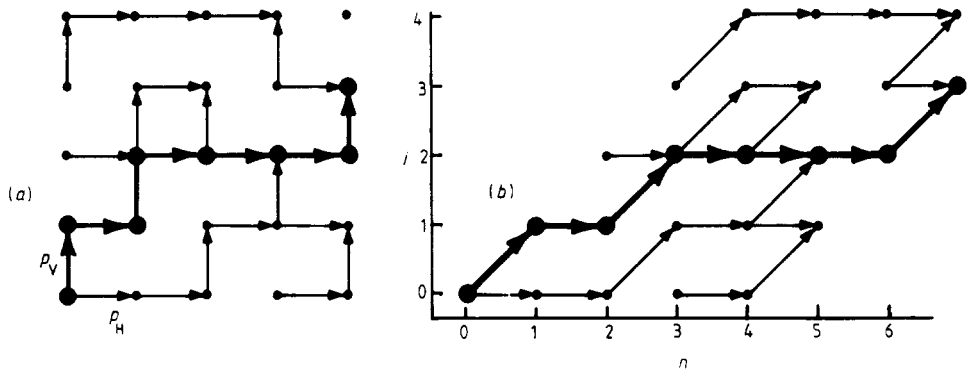


Figure 1. Lattice with $p_V = \frac{1}{2}$, $p_H = \frac{3}{4}$; (*a*) drawn as square lattice; (*b*) drawn so as to stress the coordinates i = number of vertical steps ($\triangleq$ space) and n = total number of steps ($\triangleq$ time).

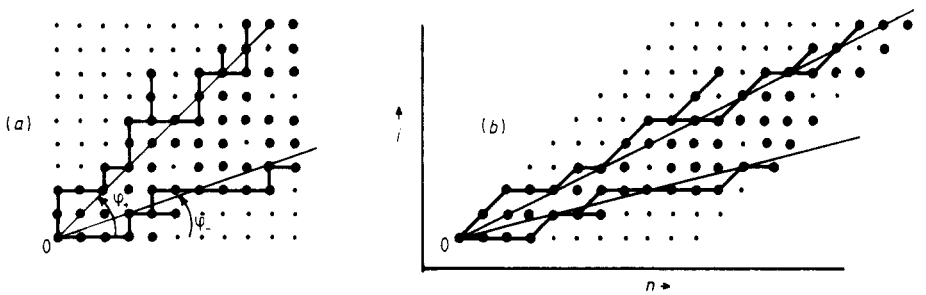


Figure 2. Percolating cluster (heavy points); its boundaries are asymptotically at $\phi = \phi_\pm$. (*b*) is the same as (*a*) but redrawn on a lattice stressing the 'time' coordinate n.

Then we can study the probabilities

$$P_n^{(+)}(i; \xi_1, \xi_2, \ldots, \xi_k) = \text{prob (front at } i; X_{i-1,n} = \xi_1, X_{i-2,n} = \xi_2, \ldots, X_{i-k,n} = \xi_k) \tag{3}$$

that the front is at i and the k sites next to it have 'occupancies' $X_{i-j} = \xi_j$. Furthermore, we shall need the probabilities

$$Q_n^{(+)}(\xi_1, \ldots, \xi_k) = \sum_i P_n^{(+)}(i; \xi_1, \ldots, \xi_k) \tag{4}$$

that $X_{i-j} = \xi_j$, irrespective of the position of the front. Probabilities $P_n^{(-)}(i; \xi_1, \ldots, \xi_k)$ and $Q_n^{(-)}(\xi_1, \ldots, \xi_k)$ referring to the occupancies $X_{i+j} = \xi_j$ of the sites close to the trailing edge are defined analogously.

Our method is based on the following two hypotheses, supposed to be valid for $p_V > p_{V,c}$:

(*A*) For any infinite cluster occupying finitely many points in the first column $n = 0$, the probabilities $Q_n^{(\pm)}(\xi_1, \ldots, \xi_k)$ tend with $n \to \infty$ towards unique limiting distributions $Q^{(\pm)}(\xi_1, \ldots, \xi_k)$.

(*B*) For small ε we can assume

$$Q_n^{(\pm)}(\xi_1, \ldots, \xi_k) = O(\varepsilon^\nu) \tag{5}$$

where

$$\nu = k - \xi_1 - \xi_2 - \cdots - \xi_k \tag{6}$$

is the number of sites next to the front (edge), with distance $\leq k$, which do *not* belong to the infinite cluster. For $n \to \infty$ this property carries over to $Q^{(\pm)}(\xi_1, \ldots, \xi_k)$.

We have no rigorous proof of property (A). We have verified it, however, to the lowest non-trivial order in ε (i.e. accepting property (B)).

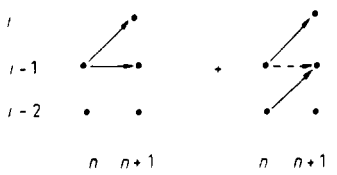
Property (B) can be proven, for finite n, by induction. Let us assume it for a certain value of n. In going from the nth to the $(n+1)$th column, the number ν of sites not belonging to the infinite cluster can increase only if (a) the vertical bonds between $(i \pm j, n)$ and $(i \pm j, n+1)$ are broken, or (b) the front recedes (trailing edge proceeds) since the bond between (i, n) and $(i, n+1)$ is broken. Since both occur only with probability ε, we thus see that property (b) holds also for $n+1$. This does not yet show that equation (5) holds for *any* infinite cluster, as it does not need to hold for $n = 0$. Due to property (A), however, it is sufficient to consider only those clusters for which it does hold for $n = 0$. Notice, that we have no rigorous proof that property (B) holds for $n \to \infty$. But again, we shall verify this perturbatively.

It is straightforward (although increasingly tedious with increasing order in ε) to set up the systems of master equations for the probabilities $P_n^{(\pm)}(i; \xi_1, \ldots, \xi_k)$. The lowest non-trivial approximation for $P_n^{(+)}(i; \xi_1)$, for example, reads

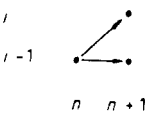
$$\begin{aligned} P_{n+1}^{(+)}(i;1) = {} & p_V[1-\varepsilon(1-p_V)]P_n^{(+)}(i-1;1) + p_V P_n^{(+)}(i-1;0) \\ & + (1-p_V)[1-2\varepsilon(1-p_V)]P_n^{(+)}(i;1) \\ & + p_V(1-p_V)P_n^{(+)}(i;0) + \varepsilon(1-p_V)^2 P_n^{(+)}(i+1;1) + O(\varepsilon^2) \end{aligned} \tag{7}$$

$$P_n^{(+)}(i;0) = \varepsilon p_V(1-p_V)P_n^{(+)}(i-1;1) + \varepsilon(1-p_V)^2 P_n^{(+)}(i;1) + (1-p_V)^2 P_n^{(+)}(i;0) + O(\varepsilon^2). \tag{8}$$

Denoting established bonds by full arrows and broken bonds by broken arrows, the first term of equation(7) is the sum of the two graphs—with weights $p_V p_H$ and $p_V^2 \varepsilon$ respectively—



The second term in equation (7) can analogously be represented as



Actually, the contribution of this graph is $p_V p_H P_n^{(+)}(i-1;0)$, but neglecting terms of order ε^2 we arrive at the term in equation (7). The other terms in equations (7) and (8) are obtained in the same way.

Master equations for $Q_n^{(\pm)}(\xi_1, \ldots, \xi_k)$ can now be obtained by summing over the position i of the front (or trailing edge). Including terms of order ε we obtain

$$\begin{aligned} Q_{n+1}^{(+)}(1) &= [1-\varepsilon(1-p_V)]Q_n^{(+)}(1) + p_V(2-p_V)Q_n^{(+)}(0) + O(\varepsilon^2) \\ Q_{n+1}^{(+)}(0) &= \varepsilon(1-p_V)Q_n^{(+)}(1) + (1-p_V)Q_n^{(+)}(0) + O(\varepsilon^2) \end{aligned} \tag{9}$$

and

$$\begin{aligned} Q_{n+1}^{(-)}(1) &= [1-\varepsilon(1-p_V)]Q_n^{(-)}(1) + p_V Q_n^{(-)}(0) + O(\varepsilon^2) \\ Q_{n+1}^{(-)}(0) &= \varepsilon(1-p_V)Q_n^{(-)}(1) + (1-p_V)Q_n^{(-)}(0) + O(\varepsilon^2). \end{aligned} \tag{10}$$

One sees that indeed both equations have stationary solutions, with

$$Q^{(+)}(0) = \varepsilon \frac{1-p_V}{p_V(2-p_V)} + O(\varepsilon^2) \tag{11}$$

$$Q^{(-)}(0) = \varepsilon \frac{1-p_V}{p_V} + O(\varepsilon^2) \tag{12}$$

and

$$Q^{(\pm)}(1) = 1 - Q^{(\pm)}(0). \tag{13}$$

In a similar way one can construct an equation governing the average front (or trailing edge) position

$$\langle i \rangle_n^{(\pm)} = \sum_i P_n^{(\pm)}(i) \cdot i = \sum_i \sum_{\xi_1, \ldots, \xi_k} i P_n^{(\pm)}(i; \xi_1, \ldots, \xi_k). \tag{14}$$

Including terms of order ε^2 we find

$$\langle i \rangle_{n+1}^{(+)} = \langle i \rangle_n^{(+)} + p_V - \varepsilon(1-p_V)^2 - \varepsilon^2(1-p_V)^3 - \varepsilon(1-p_V)Q_n^{(+)}(0) \tag{15}$$

$$\langle i \rangle_{n+1}^{(-)} = \langle i \rangle_n^{(-)} + \varepsilon + \varepsilon^2(1-p_V) + \varepsilon(1-p_V)Q_n^{(-)}(0). \tag{16}$$

Inserting equations (11) and (12), and transforming back to the original coordinates, we find

$$\tan \phi_+ = \lim_{n\to\infty} \frac{\langle i \rangle_{n+1}^{(+)} - \langle i \rangle_n^{(+)}}{1 - \langle i \rangle_{n+1}^{(+)} + \langle i \rangle_n^{(+)}} = \frac{p_V}{1-p_V} - \varepsilon - \frac{\varepsilon^2}{p_V(2-p_V)} + O(\varepsilon^3) \tag{17}$$

and similarly

$$\tan \phi_- = \varepsilon + \varepsilon^2/p_V + O(\varepsilon^3). \tag{18}$$

For $p_V = 1-\delta$ with $\delta \ll 1$ one checks that indeed

$$\tan \phi_-(\varepsilon, \delta) = [\tan \phi_+(\delta, \varepsilon) + O(\varepsilon^2)]^{-1} + O(\delta^2). \tag{19}$$

A similar analysis shows that the dispersion D_n of the position of the front increases linearly with n,

$$\begin{aligned} D_{n+1}^{(+)} &\equiv \langle i^2 \rangle_{n+1}^{(+)} - (\langle i \rangle_{n+1}^{(+)})^2 \\ &= D_n^{(+)} + p_V(1-p_V) + \varepsilon(1-p_V)^2(1+2p_V) \\ &\quad + 3\varepsilon^2(1-p_V)^2\left((1-p_V)p_V + \frac{1}{p_V(2-p_V)}\right), \end{aligned} \tag{20}$$

and similarly for the trailing edge

$$D_{n+1}^{(-)} = D_n^{(-)} + \varepsilon + \varepsilon^2\left(2 - 3p_V + \frac{3(1-p_V)^2}{p_V}\right). \tag{21}$$

This indidates clearly that the transverse growth of the cluster is essentially a random walk.

Neglecting terms of order ε^3, the threshold $p_\phi^{(+)}$ defined in the introduction is obtained from equation (17) as

$$p_\phi^{(+)} = \frac{\tan\phi}{1+\tan\phi} + \frac{\varepsilon}{(1+\tan\phi)^2} + \varepsilon^2\left(\frac{1}{\tan\phi\cdot(2+\tan\phi)} - \frac{1}{1+\tan\phi)^3}\right). \tag{22}$$

The Domany–Kinzel threshold, in particular, is

$$p_{DK} = p_{45°}^{(+)} = \tfrac{1}{2} + \tfrac{1}{4}\varepsilon + \tfrac{5}{24}\varepsilon^2. \tag{23}$$

Values of $\lambda \equiv p_{DK}/p_H$ calculated from this are plotted in figure 4 together with the results of Domany and Kinzel (1981), showing excellent agreement. In particular, we get $p_{V,c} \simeq 0.625$, in surprisingly good agreement with the value $p_c = 0.645$ of Kinzel and Yeomans (1981).

In order to calculate the critical bond probability $p_{V,c}$, we have to equate ϕ_+ and ϕ_-. Before doing this, we have to decide whether we should calculate ϕ_- from equation (18) or from equations (1) and (17). *A priori*, neither of the two methods seems preferable, since we have to use them both outside their supposed domains of validity. Plotting curves $\phi_\pm = \text{const}$, calculated from equations (17) and (18), versus p_V and p_H, we found that ϕ_- deteriorates very rapidly as soon as one leaves the domain $\varepsilon \ll p_V$. In contrast, ϕ_+ remains an excellent approximation also for large ε, provided $p_V \geq \varepsilon$.

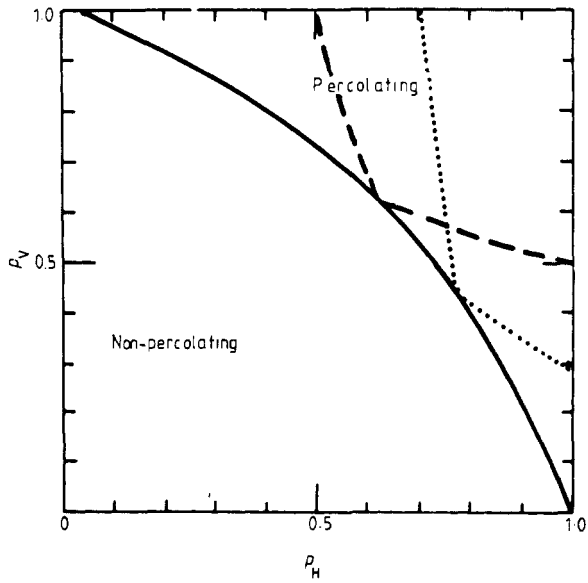


Figure 3. Percolating and non-percolating regions, separated by $p_V = p_{V,c}$ (full curve); broken line, $p_V = p_{DK} = p_{45°}^{(\pm)}$; dotted line, $p_V = p_{22.5°}^{(+)}$ (lower part) and $p_V = {}^{(-)}_{22.5°}$ (upper part).

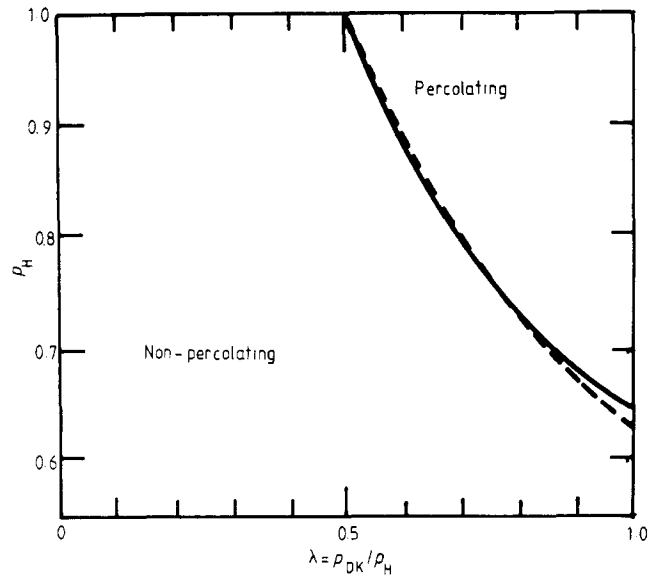


Figure 4. Domany–Kinzel threshold for percolation along $\phi = 45°$: full curve, finite-size scaling result of Domany and Kinzel (1981); broken curve, result of equation (23).

Thus we computed $p_V = p_\phi^{(+)}(p_H)$ and $p_H = p_{90°-\phi}^{(+)}(p_V)$ from equation (22). The intersection of these curves yields $p_{V,c}$. The result is shown in figure 3 together with $p_{22.5°}^{(\pm)}$ and p_{DK}. We might mention that the order ε results (computed by dropping the last term in equation (22)) are very similar, indicating a fast convergence of the expansion.

We might add that we cannot compute critical exponents, as in any perturbative calculation.

3. Discussion

(*a*) As we have already mentioned, including higher orders in ε would be extremely tedious. Except for this, however, there does not seem any problem.
(*b*) There does not seem any problem either in applying our method to other directed percolation problems in which bonds in at least one direction are unbroken with probability near to unity. One example would be the triangular lattice model studied by Wu and Stanley (1982). Another example is the model of Grassberger and de la Torre (1979). The method used in appendix B of that paper to estimate the kink velocity for small k is indeed the lowest order of the systematic approach of the present paper.
(*c*) As a generic rule, it seems that one needs the distributions $Q^{(\pm)}$ to order ε^k if one wants to compute $\phi_\pm$ to order ε^{k+1}. However, in some cases (such as the one in the appendix B mentioned) an approximation of $Q^{(\pm)}$ to lower order in ε is sufficient.
(*d*) In connection with epidemic processes (Griffeath 1979, Bailey 1975), much work has been done on the velocity of the front (corresponding to our ϕ_+) in the case of continuous time and discrete space. It seems that our method should be applicable in this case as well (see the suggestion in Mollison (1977)).

(*e*) It is unclear whether our method is applicable to higher-dimensional cases. If the front—which then is a $(d-1)$-dimensional hypersurface—would remain essentially plane for $n \to \infty$, one would expect properties (A) and (B) to continue to hold. However, if it develops long-range fluctuations, this would no longer be true.

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References

Bailey N T 1975 *The Mathematical Theory of Infectious Diseases* 2nd edn (New York: Hafner)
Blease J 1977 *J. Phys. C: Solid State Phys.* **10** 147, 3561
Broadbent S R and Hammersley J M 1957 *Proc. Camb. Phil. Soc.* **53** 629
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Dhar D and Barma M 1981 *J. Phys. A: Math. Gen.* **14** L1
Domany E and Kinzel W 1981 *Phys. Rev. Lett.* **47** 5
Grassberger P 1977 *Nucl. Phys.* B **125** 83
Grassberger P and de la Torre A 1979 *Ann. Phys. NY* **122** 373
Griffeath D 1979 *Additive and Cancellative Interacting Particle Systems, Springer Lecture Notes in Mathematics* vol 724
Kertész K and Viczek T 1980 *J. Phys. C: Solid State Phys.* **13** L343
Kinzel W and Yeomans J M 1981 *J. Phys. A: Math. Gen* **14** L164
Mollison D 1977 *J. R. Stat. Soc.* B **39** 283
Moshe M 1978 *Phys. Rep.* **37C** 255
Obukhov S P 1980 *Physica* **101A** 145
Schlögl F 1972 *Z. Phys.* **253** 147
Schulmann L S and Seiden P E 1982 *J. Stat. Phys.* **27** 83
Wu F Y and Stanley H E 1982 *Phys. Rev. Lett.* **48** 775